Solutions

1.4: Separable Equations

Definition 1. A first-order differential equation $dy/dx = f(x, y)$ is **separable** if the function f can be written as the product of two independent functions of one variable each; i.e. $f(x, y) = g(x)h(y)$. In this case, we simply separate the y's and x's and integrate them separately.

Example 1. Find the general solution for the differential equation

$$\frac{dy}{dx} = -6xy.$$

$$\int \frac{dy}{y} = \int -6x\,dx$$

$$\ln|y| = -3x^2 + C \Rightarrow y = e^{-3x^2+C} = Ce^{-3x^2}.$$

Example 2. Solve the differential equation

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}.$$

$$\int 3y^2 - 5\,dy = \int 4 - 2x\,dx$$

$$y^3 - 5y = 4x - x^2 + C$$

Hard to solve for y explicitly w/o initial conds.

Exercise 1. Find all solutions to the differential equation

$$\frac{dy}{dx} = 6x(y-1)^{2/3}.$$

$$\int (y-1)^{-2/3}\,dy = \int 6x\,dx$$

$$3(y-1)^{1/3} = 3x^2 + C$$

$$(y-1) = (x^2+C)^3$$

$$y = (x^2+C)^3 + 1$$

Exercise 2. Find all solutions to the differential equation

$$2\sqrt{x}\frac{dy}{dx} = \cos^2 y, \quad y(4) = \pi/4.$$

$$\int \sec^2 y\,dy = \int \frac{1}{2\sqrt{x}}\,dx$$

$$\tan y = \sqrt{x} + C$$

$$y = \tan^{-1}(\sqrt{x} + C)$$

$$y(4) = \pi/4 = \tan^{-1}(2 + C) \Rightarrow C = -1$$

$$y = \tan^{-1}(\sqrt{x} - 1)$$

Natural Growth and Decay. The differential equation

$$\frac{dx}{dt} = kx \quad (k \text{ is a constant}) \tag{1}$$

serves as a mathematical model for a wide variety of natural phenomena, such as, population growth, compound interest, radioactive decay or drug elimination to name a few. Solve the differential equation given in (1).

$$\int \frac{dx}{x} = \int k\,dt$$

$$\ln|x| = kt + C$$

$$x = e^{kt+C} = Ce^{kt}$$

Example 3. A specimen of charcoal found at Stonehenge turns out to contain 63% as much ^{14}C as a sample of present-day charcoal of equal mass. Given that the half-life of ^{14}C is ~~[illegible]~~ 5700 years, we can solve for the constant $k \approx 0.0001216$. What is the age of the sample?

$$(0.63) = e^{-kt} \Rightarrow t = -\frac{\ln(0.63)}{0.0001216} \approx 3800 \text{ years}$$

Example 4. Recall that Newton's law of cooling is given by the differential equation

$$\frac{dT}{dt} = k(A - T)$$

for a positive constant k, where $T(t)$ is the temperature of a body immersed in a medium of constant temperature A. Consider the following: A 4-lb roast, initially at 50°F, is placed in a 375°F oven at 5:00pm. After 75 minutes it is found that the temperature $T(t)$ of the roast is 125°F. When will the roast be 150°F (medium rare)?

$T_0 = 50°, \quad T(75) = 125°$

$$\int \frac{dT}{375 - T} = \int k\,dt$$

$$-\ln(375 - T) = kt + C$$

$$T = 375 - Ce^{-kt}$$

$T_0 = 50° \Rightarrow C = 325$

$\Rightarrow k = -\frac{1}{75}\ln\left(\frac{250}{325}\right) \approx 0.0035$

So $T = 375 - 325e^{-0.0035t}$

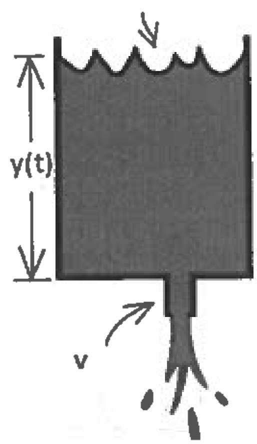
$t \approx 105$ so remove at 6:45pm

Torricelli's Law. Suppose that a water tank has a hole with area a at the bottom. Let $y(t)$ represent the depth of water and $V(t)$ represent the volume of water in the tank. It is true, under ideal conditions, that the velocity of water exiting through the hole is

$$v = c\sqrt{2gy}.$$

(We take $c = 1$ for simplicity.) We arrive at the equation

$$\frac{dV}{dt} = -av \quad \text{or equivalently} \quad A(y)\frac{dy}{dt} = -av.$$



Exercise 3. A hemispherical bowl has to radius 4 ft and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 in. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?

$$A(y) = \pi r^2 = \pi\left[16 - (4-y)^2\right] = \pi(8y - y^2)$$

With $g = 32\ \text{ft}/\text{s}^2$,

$$\pi(8y - y^2)\frac{dy}{dt} = -\pi\left(\frac{1}{24}\right)^2\sqrt{2\cdot 32y}$$

$$\int 8y^{1/2} - y^{3/2}\,dy = -\int \frac{1}{72}\,dt$$

$$\frac{16}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -\frac{1}{72}t + C$$

$$C = \frac{448}{15} \implies t \approx 2150 \text{ seconds about } 36 \text{ min.}$$

Homework. 1-15, 19-23, 33-45, 49-61 (odd)